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Lattice-dimensionality dependence in the spherical model

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Abstract. The dependence on lattice dimensionality d of various critical-point parameters for the spherical model is studied by elementary means. In particular we show that the critical temperature $T_c(d)$ obeys the relation $T_c(d) \leqslant T_c(d+1)$ and that the gap exponent $\Delta(d)$ and susceptibility exponent $\gamma(d)$ are connected by $\Delta(d) = \gamma(d)+\frac{1}{2}$.

1. Introduction

The lattice-dimensionality (d) dependence of the critical-point behaviour of the Berlin–Kac (1952) spherical model can be studied in depth by means of quite sophisticated and diverse techniques (Joyce 1972, Milošević and Stanley 1971, Wilson 1972). The purpose of the present note is to show that considerable insight into the d dependence of this model can be derived quite simply and directly.

For the d dimensional simple cubic spherical model it has been shown (Berlin and Kac 1952) that the zero-field susceptibility χ and the temperature T are related by

$$\chi = \frac{\mu_0^2}{2J(z-d)}$$
$$\frac{2J}{k_B T} = (2\pi)^{-d}\int_0^{2\pi}\cdots\int\prod_{j=1}^{d}\mathrm{d}\omega_j\left(z-\sum_{k=1}^{d}\cos\omega_k\right)^{-1}. \tag{1}$$

Here μ_0 and J are the magnetic moment and exchange parameters, respectively, and k_B is Boltzmann's constant. It has been pointed out that except for the $d = 1$ case the resultant expressions cannot be simply inverted to yield $\chi(T)$. On the other hand, the numerical function $T(\chi)$ contains all the same information and has the virtue that its properties can be studied quite simply.

2. Theory

With this in mind, we begin by rewriting equation (1) in the form

$$X = \left(\frac{2}{\pi}\right)^d\int_0^{\pi/2}\cdots\int\prod_{j=1}^{d}\mathrm{d}\omega_j\left(x+\sum_{k=1}^{d}\cos^2\omega_k\right)^{-1}, \tag{2}$$

where $x \equiv \mu_0^2/4J\chi$ and $X \equiv 4J/k_B T$. We seek the behaviour of X as a function of x for

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given d and consequently denote the left hand side of equation (2) by $X(\varkappa\,; d)$. This is to be contrasted to what one would ultimately like, $\varkappa(X\,; d)$. Equation (2) can now be rewritten as the recursive integral equation

$$X(\varkappa\,; d+1) = \frac{2}{\pi}\int_0^{\pi/2} \mathrm{d}\omega X(\varkappa+\cos^2\omega\,; d), \tag{3}$$

whose solution is of course just given by the right hand side of equation (2) ($X(\varkappa\,; 0) \equiv 1/\varkappa$). When $\chi^{-1} = 0$, $\varkappa = 0$, and we denote the quantity $X(0; d)$ by $X_c(d)$(eg $4J/k_BT_c(d)$).

From equation (2) it is straightforward to show that $X(\varkappa; d)$ has the following properties:

$$X(\varkappa; d) \geqslant 0, \tag{4a}$$

$$\mathrm{d}X(\varkappa; d)/\mathrm{d}\varkappa \leqslant 0, \tag{4b}$$

$$\mathrm{d}^2X(\varkappa; d)/\mathrm{d}\varkappa^2 \geqslant 0, \qquad \text{etc.} \tag{4c}$$

Equation (4) then implies for any $\delta \geqslant 0$, that $X(\varkappa; d) \geqslant X(\varkappa+\delta; d)$. Substituting this into equation (3) immediately yields the results

$$X(\varkappa; d) \geqslant X(\varkappa; d+1) \tag{5a}$$

$$X(\varkappa; d+1) \geqslant X(\varkappa+1; d). \tag{5b}$$

When $\varkappa = 0$, equation (5*a*) implies that $X_c(d) \geqslant X_c(d+1)$, that is

$$T_c(d) \leqslant T_c(d+1). \tag{6}$$

A graphical study of the consequence of equations (4) and (5) shows that equation (5*a*) is equivalent to the statement that $\varkappa(X\,; d) \geqslant \varkappa(X\,; d+1)$, that is, for any T

$$\chi(T\,; d) \leqslant \chi(T\,; d+1). \tag{7}$$

If we write $\lim_{d\to\infty} X(\varkappa; d) \equiv X(\varkappa)$, equation (3) then reduces to a straightforward integral equation whose solution is just $X(\varkappa) =$ constant. However, it follows directly from equation (2) that as $\varkappa \to \infty$, $X(\varkappa; d) \to 1/\varkappa$, so that the constant introduced above must be taken as zero. This result is simply interpreted as the statement that as $d \to \infty$, $T_c(d) \to \infty$ and hence the model loses any physical content.

It is also straightforward to show that equations (3) and (4*c*) imply

$$\left|\frac{\mathrm{d}}{\mathrm{d}\varkappa}X(\varkappa; d+1)\right| \leqslant \left|\frac{\mathrm{d}}{\mathrm{d}\varkappa}X(\varkappa; d)\right|. \tag{8}$$

As we approach the critical temperature, the susceptibility $\chi(T; d)$ diverges with the power $\gamma(d)$, that is, as $\varkappa \to 0$

$$X(\varkappa; d) \sim X_c(d) - a(d)\varkappa^{1/\gamma(d)}, \tag{9}$$

where $a(d)$ is a positive quantity. Equation (8) then reduces to

$$\frac{a(d+1)}{\gamma(d+1)}\varkappa^{1/\gamma(d+1)-1} \leqslant \frac{a(d)}{\gamma(d)}\varkappa^{1/\gamma(d)-1}, \tag{10}$$

which implies that

$$\gamma(d+1) \leqslant \gamma(d). \tag{11}$$

However, it is known (Joyce 1972) that $\gamma(3) = 2$, $\gamma(d \geqslant 4) = 1$. Hence equation (10) yields the result

$$a(d) \geqslant a(d+1), \qquad d \geqslant 4. \tag{12}$$

Finally we note that Milošević and Stanley (1971) have shown that the fourth derivative of the free energy Ψ with respect to the external magnetic field H, evaluated at $H = 0$, is given by

$$\left(\frac{\partial^4 \Psi}{\partial H^4}\right)_{H=0} = -\frac{3\mu_0^4}{2J^2 k_B T}\frac{1}{(z-d)^4}\left(\frac{\int_0^{2\pi}\cdots\int \Pi_{j=1}^d \mathrm{d}\omega_j}{(z-\Sigma_{k=1}^d \cos\omega_k)^2}\right)^{-1}, \tag{13}$$

which in terms of the present notation reads

$$\left(\frac{\partial^4 \Psi}{\partial H^4}\right)_{H=0} = \frac{3\mu_0^4}{32J^3(2\pi)^d}\frac{X(\varkappa;d)}{\varkappa^4\,\mathrm{d}X(\varkappa;d)/\mathrm{d}\varkappa}. \tag{14}$$

As $T \to T_c$, $\varkappa \to 0$, and equation (14) becomes

$$\left(\frac{\partial^4 \Psi}{\partial H^4}\right)_{H=0} \sim \frac{1}{\varkappa^{3+1/\gamma(d)}}. \tag{15}$$

However, according to the definition of the gap exponent, $\Delta(d)$, $(\partial^4\Psi/\partial H^4)_{H=0}$ should have an asymptotic behaviour of the form

$$(\partial^4\Psi/\partial H^4)_{H=0} \sim (T-T_c)^{-\gamma-2\Delta} \sim \varkappa^{-(1+2\Delta(d)/\gamma(d))}. \tag{16}$$

Comparing this with equation (15), yields the result

$$\Delta(d) = \tfrac{1}{2}+\gamma(d).$$

Substitution of the known values of $\gamma(d)$ gives the Milošević and Stanley (1971) estimates of Δ.

3. Summary

The dependence of various critical-point parameters on lattice dimensionality has been studied by elementary means. In particular we have seen that the critical temperature is a non-decreasing function of d, starting with the values $T_c(1) = T_c(2) = 0$, $T_c(3) = 3{\cdot}96J/k_B$, ($T_c(4) \simeq 6{\cdot}5J/k_B$) and increases without limit. It is also shown that the gap exponent Δ and the susceptibility exponent γ are simply related, namely $\Delta(d) = \frac{1}{2}+\gamma(d)$. Although most of the results we have presented were already known, we feel that the present work gives one added insight into their lattice-dimensionality dependence.

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